IOSR Journal of Electrical and Electronics Engineering (IOSR-JEEE)
e-ISSN: 2278-1676,p-ISSN: 2320-3331, Volume 11, Issue 4 Ver. IV (Jul. – Aug. 2016), PP 52-58
www.iosrjournals.org

Nonlinear Dynamics of a Quantum Mesoscopic Circuit

H. Torres-Silva [1], D. Torres Cabezas[2]
[1.] *Escuela de Ingeniería Eléctrica y Electrónica, Universidad de Tarapacá, Arica, Casilla 6-D, Chile.*
[2] *Departamento Tecnologías de Información. Dirección del Trabajo, Agustinas 1253, Of. 509. Santiago, Chile*

Abstract: *In this article we study the nonlinear dynamics of a quantum mesoscopic circuit in the range where the initial condition* θ_0 *,that is, the initial normalized magnetic flux of the system which oscillate in the interval [*$-\theta_0$, $+\theta_0$*] is close to* $\pm\pi$*. This circuit is modeled as a LC circuit with quantized electric charge excited by energy battery that can produce an electrical discreteness charge in LC quantum circuit in the form of narrow pulses.*
Keywords: *charge discreteness, mesoscopic , quantum circuit*

I. Introduction

Let us first start by reviewing the essentials of quantum LC circuits with continuous charge, as discussed by Louisell [1]. Recall that a classical LC circuit is described by the Harmonic oscillator Hamiltonian:

$$H = \frac{\Phi^2}{2L} + \frac{Q^2}{2C} \tag{1}$$

in which Q represents the electric charge, and Φ the magnetic flux, its canonical conjugate variable; the equations of motion are the classical Hamiltonian equations. To quantize the system, the variables Q and Φ are replaced by the operators $\hat{Q}$ and $\hat{\Phi}$. Following [2, 3,4], to recover the quantization of charge within this electrical circuit approach, we introduce the replacement

$$\hat{\Phi} = \frac{2\hbar}{q_e}\sin(\frac{q_e}{2\hbar}\hat{\phi}), \tag{2}$$

where the canonical variables Q and Φ, and the Hamiltonian H become operators, $Q \to \hat{Q}$, $\Phi \to \hat{\Phi}$, $H \to \hat{H}$, and the $\hat{Q}$ and $\hat{\Phi}$ operators satisfy the canonical commutation rules $\left[\hat{Q}, \hat{\Phi}\right] = i\hbar$. In the classical case, the equations of motion are the Hamilton equations, $dQ/dt = \partial_\Phi H$ and $d\Phi/dt = -\partial_Q H$; while in the quantum case, the equations of motion are the Heisenberg equations, $\hbar d\hat{Q}/dt = -\left[\hat{Q}, \hat{H}\right]$ and $\hbar d\hat{\Phi}/dt = -\left[\hat{\Phi}, \hat{H}\right]$. The mathematical problem to be solved becomes that of finding the eigenvalues and eigenstates of the Hamiltonian operator, $\hat{H}$; to do that, one usually adopts the so-called charge representation (Q-representation), in which $\hat{Q} \to Q$ and $\hat{\Phi} \to -i\hbar d/dQ$, the Hamiltonian operator becomes

$$\hat{H} = \frac{2\hbar^2}{q_e^2 L}\sin(\frac{q_e\hat{\phi}}{2\hbar}) + \frac{\hat{Q}^2}{2C} \tag{3}$$

We treat our mesoscopic system as a quantum electrical circuit, with quantized charge.where q_e is the quantum of charge. We remark that, if charge discreteness is neglected, the operator $\hat{\Phi}$ may be directly identified with the magnetic flux operator, and therefore directly related to the current; however, when one introduces charge discreteness via the replacement above, the simple relation to the current is lost, therefore, after replacement (2) the flux operator becomes the pseudo-flux. This pseudo-flux operator satisfies the usual commutation relation $\left[\hat{Q}, \hat{\Phi}\right] = i\hbar$. Notice that, given the complexity of dealing with an operator such as the one above (2), it is simpler to work in the so-called pseudo-flux representation, in which the operator $\hat{\Phi}$ is replaced by its eigenvalue ϕ , while the charge operator is given by $\hat{Q} = i\hbar\frac{\partial}{\partial\phi}$. In this way, the resulting Hamiltonian is given by (3).

The Hamiltonian operator above constitutes our starting point, and our working hypothesis. The parameters of our theory, particularly L and C, are related to the geometry of the system, but are hard to compute for a given experimental system, therefore, they should be deduced from experimental observations. In section 2 we give a background of a semiclassical study of LC circuit. In section 3 we discuss solutions of the differential equation for the mesoscopic circuit in nonlinear regimen. Finally we give our conclusions.

II. Semiclassical Study of LC Circuit

The electrical engineer use simplified model provided by thecircuit description of a system, when compared with the more complete, however, we describe the behaviour of electrons modern circuits using the basic laws (Kirchhoff) as in a classical circuit. Many examples can be given: flux quantization on superconductors, conductance oscillations, quantum hall effects (integer and fractional), persistent currents and so on.

It would be very useful to find out to what extent a circuit-like description could be of use for the very small electronic circuits of mesoscopic devices, and what may be retained from it [5-9], for example, one area in which the "quantum LC circuit" may give valid results is in the calculation of energy spectra since it still required to solve the Schrodinger equation.

Now, we propose to go one step further in our simplification, by proposing to use a "semiclassical" approach [2, 3,4] . We start from our Hamiltonian, for the LC circuit, with quantized electric charge

$$H = \frac{2\hbar^2}{q_e^2 L}\sin\left(\frac{q_e\phi}{2\hbar}\right) + \frac{Q^2}{2C} \qquad (4)$$

and the Heisenberg motion equations are

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = -\dot{\phi}, \qquad \frac{\partial H}{\partial \phi} = \dot{Q} = \frac{\hbar}{q_e L}\sin\left(\frac{q_e\phi}{\hbar}\right) \qquad (5)$$

This pair of equations may be written as a single second order equation

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0 \qquad (6)$$

where we define $\theta = \phi/\phi_0$, with $\phi_0 = \hbar/q_e$, (ϕ and Q are c-numbers), $\omega_0^2 = 1/LC$. Equation (6) is the same expression than simple pendulum, a system that has been quantized by several authors [5], (see references therein). Discrete-charge quantum circuit have been studied other authors [10-13], to obtain approximate quantum energy eigenstates for initial condition $\sin\theta_0 \approx \theta_0$ (radians). In this paper we will study the nonlinear dynamical behavior when $\theta_0 \le \pi$.

At the initial instance, the energy of the mesoscopic circuit is the sum of the kinetic and potential energy, from (4) we obtain :

$$E_0 = \dot{\theta}_0^2 + E_{LC}\sin^2(\theta_0/2) \text{ or}$$

$$E_0/2 = \dot{\theta}_0^2/2 + (E_{LC}/2)\sin^2(\theta/2) \qquad (7)$$

with $E_{LC} = 4\omega_0^2 = 4/LC$ which has been normalized and $\phi/\phi_0 = \theta,\ \hbar/q_e = \phi_0$. Because of the assumed lossless system , the initial energy is preserved for all instances:

$$E_0 = \dot{\theta}^2(t) + E_{LC}\sin^2(\theta(t)/2)$$

Here, $E_p/2 = 2\omega_0^2$ is the maximum possible (normalized) potential energy of the quantum circuit, being attained when $\dot{\theta} = 0$. Here, we are interested in the quantum circuit when $E_0 \le E_{LC}$ where

$$\dot{\theta}^2(t) = E_0 - E_{LC}\sin^2(\theta(t)/2) \geq E_0 - E_{LC} < 0 \qquad (8)$$

III. Solution of the Differential Equation of Motion for the Mesoscopic Circuit when $E_0 \leq E_{LC}$

Analytically to determine the dynamics of the mesoscopic circuit, we must solve the equation

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0$$

This is similar to the approach proposed in the work of [14] for pendulum theory, but will impose an initial condition where $\dot{\theta}_0$ is zero. That is to say,

$$\theta(t=0) = \theta_0, \quad \left.\frac{d\theta}{dt}\right|_{t=0} = \dot{\theta}_0 = 0 \qquad (9)$$

In this equation , θ_0 is the initial angular displacement of the system which oscillates in the interval [- θ_0 , + θ_0] . Now we multiply equation (6) by $\frac{d\theta}{dt}$, we have

Which is written as

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta\right] = 0 \qquad (10)$$

Equation (10) refers to the conservation of the total mechanical energy, to integrate and apply the initial conditions given by equation (9), we obtain

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2\left[\cos(\theta(t)) - \cos(\theta_0(t))\right] \qquad (11)$$

The solution of this equation is given in appendix A

$$\theta(t) = 2\sin^{-1}\left[k\,\mathrm{sn}(F(\sin^{-1}(1),k) \pm \omega_0 t, k\right] \qquad (12)$$

(See appendix A), therefore equation (12) reproduces the known formula for the nonlinear pendulum [14].

$$\theta(t) = 2\sin^{-1}\left[\sqrt{k}\,\mathrm{sn}(F(\sin^{-1}(1),k) \pm \omega_0 t, k\right] \qquad (13)$$

However $F(\frac{\pi}{2},k) = K(k)$ is the first class elliptical integral [14], so

$$\theta(t) = 2\sin^{-1}\left[\sqrt{k}\,\mathrm{sn}(K(k) \pm \omega_0 t, k\right] \qquad (14)$$

The period T is the time required to complete a cycle, in this case, the oscillation period T is four times the time taken from $\theta = 0, (\lambda = 0)$, $\theta = \theta_0, (\lambda = 1)$ (see appendix A) so

$$\omega_0 / \omega = 4K(k) \qquad (15)$$

In Figure 1, the period T differs appreciably from T_0 only for large amplitudes. For small amplitudes it is sufficient to take the first corrective term, sufficient approximation for most practical situations as is done by other authors [12-14].

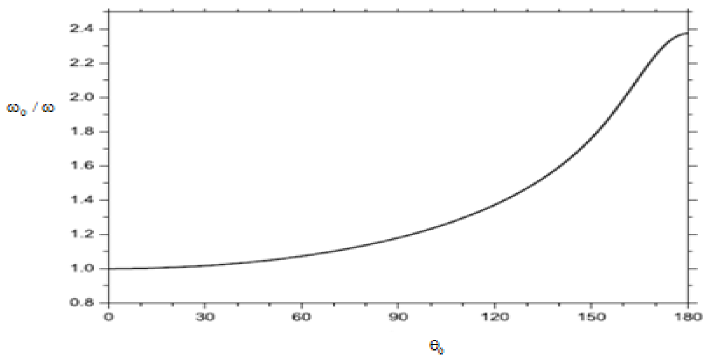

Figure 1. Change in the frequency or period of mesoscopic circuit according to the amplitude, equation (15).

As the initial angular displacement increases (and its initial angular velocity), the oscillation period $\omega_0 / \omega = T / T_0$ increases considerably. That is, the movement becomes disharmonious. Since the magnitude of $\sin\theta$ is always less than the angle θ, the acceleration of real circuit will be noticeably lower than $\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta$ (valid for small angles where $\sin\theta \approx \theta$), and corresponds to the simple angular harmonic motion. For this reason, once the movement becomes anharmonic significantly, the period no longer remains constant but lengthens increasingly with increasing amplitude.

Now to study the nonlinear behavior of the mesoscopic circuit we suppose a normalized battery energy $E_B = E_{real} / 2C\phi_0^2\omega_0^2$ which can be positive or negative that is connected to LC circuit. We only consider the adiabatic approximation so that battery energy $E_B(t)$ is consider as a constant . Actually, the linear term given by the battery can be moved by a translation in the 'coordinate' (charge) space and we make a re-definition of $\dot{Q}$ and Q and a shift of the energy E_0.

The condition energy conservation is

$$\dot{\theta}^2(t) = E_0 - E_{LC}\sin^2(\theta(t)/2) \geq E_0 - E_{LC} < 0$$

Here, E_B is included in E_0. According equation (11)

$$(\frac{d\theta}{dt}) = \sqrt{2\omega_0^2\left[\cos(\theta(t)) - \cos(\theta_0(t))\right]} \qquad (16)$$

When $\theta(t) \approx \pi$, $\dot{\theta}_0 = 0$, the critical angular velocity is

$$(\frac{d\theta}{dt})_c = \pm\omega_0\sqrt{2\left[1 + \cos\theta_0\right]} \qquad (17)$$

where

$$\theta(t) = \theta(t - T),\ \dot{\theta}(t) = \dot{\theta}(t - T),\ T = 4K(k)/\omega_0.$$

If the mesoscopic circuit has solely the initial condition θ_0 without any initial angular velocity $\dot{\theta}_0$, $\dot{\theta}_0 = 0$, then we have $k = 1/2$. For these special initial values and the initial instance

$$\omega t = \omega_0 t + K(k)$$
$$\theta(t) = 2\arcsin(k\,\mathrm{sn}(\omega t, k)) \qquad (18)$$
$$\dot{\theta}(t) = \sqrt{E_0}\,\mathrm{cn}(k\,\mathrm{sn}(\omega t, k))$$

which is the known solution in the literature [10] for nonlinear pendulum. . Numerical values are given for different values of E_B but with $0 < \theta_0 < \pi$.

For amplitudes shown in Table 1, the harmonic function provides an excellent approximation to the periodic solution of Eq. (18) and the periodic motion exhibited by a simple mesoscopic circuit is practically harmonic but its oscillations are not isochronous (the period is a function of the amplitude of oscillations).

From table 1, we can conclude that for initial amplitudes as high as 0.75π the effect of the nonlinearity is seen only in the fact that the frequency of the oscillation $\dot{\theta}$ depends on the amplitude θ_0 of the motion.

Table 1

max θ	min θ	$\dot{\theta}$	θ_0	$\omega_0 t \times 10^{-8}$	E_B
0.3	-0.3	0	0.1 π	2, 4.8, 8, 11, 14	-0.4
0.75	-0.75	0	0.25 π	2, 5, 8.2, 11.3,14.5	-0.25
1.5	-1.5	0	0.5 π	2.1, 5.5 , 9,13 ,16.5	0
2.5	-2.55	0	0.75 π	2.4, 7, 12,16.3,22.5	0.25
3.0	-3.0	0	0.99 π	5. 5, 17 , ...	0.5

Table 1, shows values of different parameters of the mesoscopic circuit for $\dot{\theta}=0$ at microwave frequency obtained from equation (18).

As a comparison between Figures 2 and 3, Figure 1 shows a temporal variation of θ practically harmonic while Figure 3 presents a large period with $\dot{\theta}$ (dashed line) as a set of pulses corresponding to a discrete charge configuration.

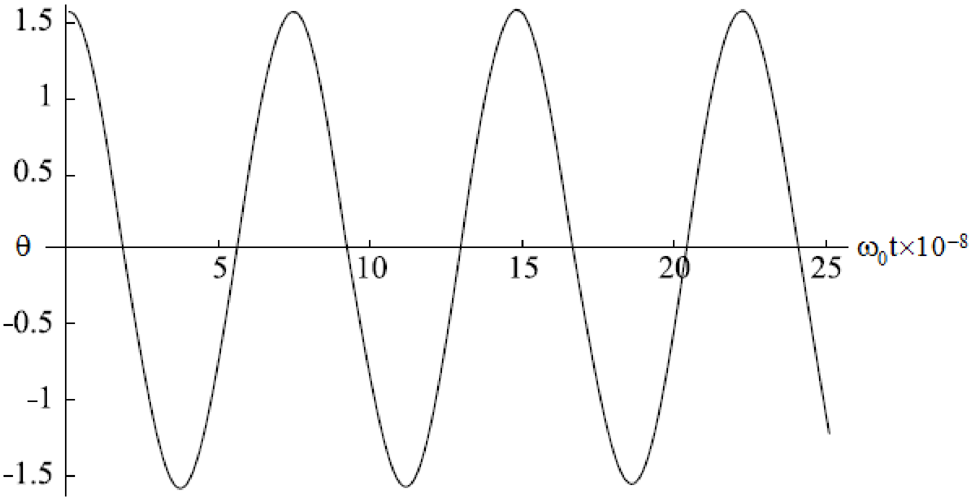

Figure 2 Shows an example for the solution of the amplitude θ with $\theta_0=0.5\pi$ and $E_B=0$.

As has been stated above, sn and cn are periodic functions having a real period equal to four times the complete elliptic integral of first kind. Therefore, the amplitude θ is also a periodic function.

Figure 3, shows the amplitude θ and the angular velocity $\dot{\theta}$. They are also periodic functions for $\theta_0=0.995\pi$, : $E_B=0.5$. Therefore, the amplitude θ and the angular velocity are also periodic functions. However the dynamics of the circuit is is highly nonlinear.

Here $\dot{\theta}$ is proportional to the discreteness charge Q of the quantum circuit $\frac{Q}{C}=-\dot{\phi}\rightarrow\frac{Q}{C\phi_0}=-\dot{\theta}$ (19)

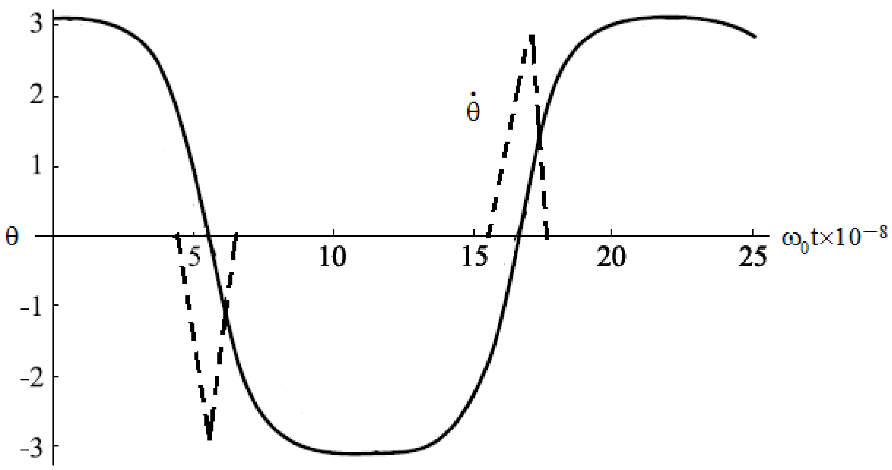

Figure 3, the amplitude θ and the angular velocity are also periodic functions for $\theta_0=0.995\pi$, : $E_B=0.5$.

IV. Conclusion

In this article we have studied the nonlinear dynamics of a quantum mesoscopic circuit in the range where the initial condition θ_0 that is the initial normalizes magnetic flux of the system which oscillate in the interval [- θ_0 , $+\theta_0$] is close to $\pi \pm$. This circuit is modeled as a LC circuit with quantized electric charge excited by energy battery that can produce an electrical charge discreteness in LC quantum circuit in the form of narrow pulses.

In the event of small oscillations, the energy E_0 of the circuit is small compared to the maximum possible potential energy E_{LC} . This leads to a modulus close to zero, for which the Jacobi elliptic functions can be replaced by trigonometric functions.

The period of oscillation of the mesoscopic circuit is constant and independent of the initial angular displacement for values of $\theta_0 = \pi / 2$, as shown in Figure 1. As the initial angular displacement increases (and its initial angular velocity), the oscillation period increases considerably when $\theta_0 \approx \pi$. That is, the movement becomes disharmonious. For this reason, once the movement becomes dissonant or anharmonic significantly, the period no longer remains constant but lengthens increasingly with increasing amplitude. The results presented also intend to bring the student of physics and engineering , the introduction of elliptic integrals and motivate the search for new alternatives offered programming to solve physical and applied mathematical problems.

References

[1] W.H. Louisell, Quantum Statistical Properties of Radiation, Wiley, USA, 1973.
[2] Y.Q. Li, B. Chen, Phys. Rev. B 53 (1996) 4027.
[3] C.A. Utreras-Díaz, D. Laroze, Mod. Phys. Lett. B 26 (2012) 1250138.
[4] C. A. Contreras and D. Laroze, Physica B, 476, 77-81, (2015).
[5] S. Datta, Lessons from Nanoelectronics, a Lecture Notes Series (Volume 1, Publication, Singapore :World Scientific Press, 2012).
[6] S. Lyshevski, Nano and Molecular Electronics Handbook, CRC Press. Taylor & Francis Group, 2016.
[7] H. J. de Los Santos, Principles and Applications of Nano MEMS Physics. Springer, Berlin, 2005.
[8] David K. Ferry and Stephen M. Goodnick, Transport in Nanostructures. (Cambridge University Press, 2009).
[9] W. D. Heiss (Ed.), Quantum Dots: Doorway to Nanoscales Physics. (Springer, Berlin 2005).
[10] B. Chen et al, Phys. Lett. A 335, 103 (2005), and references therein.
[11] J. C. Flores, Europhys. Lett. 69, 116 (2005). .
[12] J. C. Flores et al, Phys. Rev. B 74, 193319, (2006). and references therein**..**
[13] C. Utreras, Physics Letters A, 372, 5059, (2008).
[14] Belendez A, Pascual C, Mendez D I, Belendez T and Neipp C 2007 Exact solution for the non-linear pendulum, *Rev. Bras. Ens. Fis.* **29** 645–8, and references therein
[15] [M. Abramovitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, 10th ed.,National Bureau of Standards, USA, 1972.

Appendix A

Analytically to determine the dynamics of the mesoscopic circuit, we must solve the equation

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0 \qquad \text{(A0)}$$

This is similar to the approach proposed in the work of [10] for nonlinear pendulum theory, but will impose another initial condition, that is the initial angular velocity $\dot{\theta}_0$ is zero. That is to say,

$$\theta(t=0) = \theta_0 , \quad \left.\frac{d\theta}{dt}\right|_{t=0} = \dot{\theta}_0 = 0 \qquad \text{(A1)}$$

In this equation , $\theta_0 = \dfrac{\phi(t=0)}{\phi_0}$ is the initial angular displacement of the system which oscillate in the interval [- θ_0 , $+\theta_0$] and $\dot{\theta}_0$ is the initial angular velocity. Now we multiply equation (A0) by $\dfrac{d\theta}{dt}$

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\omega_0^2 \sin\theta$$

Which is written as

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2 \cos\theta\right] = 0 \qquad \text{(A2)}$$

Equation (A2) refers to the conservation of the total mechanical energy, to integrate and apply the initial conditions given by equation (A1), we obtain

$$(\frac{d\theta}{dt})^2 = 2\omega_0^2\left[\cos(\theta(t)) - \cos(\theta_0(t))\right] \qquad \text{(A3)}$$

Given the trigonometric identity

$$\cos\theta = 1 - 2\sin^2(\frac{\theta}{2})$$

in equation (1) and considering the following changes of variables:

$$\sin^2(\frac{\theta_0}{2}) = k\,, \quad \sin(\frac{\theta}{2}) = \zeta(t) \qquad \text{(A4)}$$

Equation (A3) in terms of the new variables is given by

$$(\frac{d\theta}{dt})^2 = 4\omega_0^2\left[k - \zeta^2\right] \qquad \text{(A5)}$$

where the initial conditions given by equation (A1), satisfy

$$\zeta(t=0) = \sqrt{k}\,, \quad \text{and} \quad \frac{d\zeta}{dt} = \frac{d\zeta}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}\cos(\frac{\theta}{2})\frac{d\theta}{dt} \qquad \text{(A6)}$$

It should be noted that θ varies between 0 and π, therefore, it is determined k in 0 <k <1. From equations (A5) and (A6) follows

$$(\frac{d\zeta}{dt})^2 = \omega_0^2 k\left[1 - \zeta^2\right](1 - \frac{\zeta^2}{k}) \qquad \text{(A7)}$$

effecting a change of dimensionless variables, defined by:

$$\tau = \omega_0 t\,, \quad \lambda = \frac{\zeta}{\sqrt{k}} \qquad \text{(A8)}$$

Equation (A5) in terms of the new variables is:

$$(\frac{d\lambda}{d\tau})^2 = \left[1 - k\lambda^2\right](1 - \lambda^2) \qquad \text{(A9)}$$

where $\lambda(\tau = 0) = 1$, (A10)

From equation (15) we have

$$d\tau = \pm\frac{d\lambda}{\sqrt{\left[1 - k\lambda^2\right](1 - \lambda^2)}} \qquad \text{(A11)}$$

integrating both sides of equation (A11) we get easily

$$\tau = \pm\frac{1}{\sqrt{k}}\left[-\int_0^1 \frac{d\lambda}{\sqrt{\left[1/k - \lambda^2\right](1 - \lambda^2)}} + \int_o^\lambda \frac{d\lambda}{\sqrt{\left[1/k - \lambda^2\right](1 - \lambda^2)}}\right] \qquad \text{(A12)}$$

Incomplete elliptic integrals of first class are defined as (Abramowitz and Stegun, 1972): each of the integral equation (A12) are expressed in the form,

$$\frac{1}{\sqrt{k}}\int_o^z \frac{d\lambda}{\sqrt{\left[1/k - \lambda^2\right](1 - \lambda^2)}} = \mathrm{sn}^{-1}(\lambda, k) \qquad \text{(A13)}$$

$$\frac{1}{\sqrt{k}}\int_o^1 \frac{d\lambda}{\sqrt{\left[1/k - \lambda^2\right](1 - \lambda^2)}} = F(\sin^{-1}(1), k) \qquad \text{(A14)}$$

Then equation (A12) is expressed as:

$$\pm\tau(\lambda) = \mathrm{sn}^{-1}(\lambda, k) - F(\sin^{-1}(1), k) \qquad \text{(A15)}$$

Finally we can obtain the angular displacement, in equation (A15) as

$$\theta(t) = 2\sin^{-1}\left[k\,\mathrm{sn}(F(\sin^{-1}(1), k) \pm \omega_0 t, k\right] \qquad \text{(A16)}$$